

General d -position sets

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Abstract

The general d -position number $\text{gp}_d(G)$ of a graph G is the cardinality of a largest set S for which no three distinct vertices from S lie on a common geodesic of length at most d . This new graph parameter generalizes the well studied general position number. We first give some results concerning the monotonic behavior of $\text{gp}_d(G)$ with respect to the suitable values of d . We show that the decision problem concerning finding $\text{gp}_d(G)$ is NP-complete for any value of d . The value of $\text{gp}_d(G)$ when G is a path or a cycle is computed and a structural characterization of general d -position sets is shown. Moreover, we present some relationships with other topics including strong resolving graphs and dissociation sets. We finish our exposition by proving that $\text{gp}_d(G)$ is infinite whenever G is an infinite graph and d is a finite integer.

Keywords: general d -position sets; dissociation sets; strong resolving graphs; computational complexity; infinite graphs

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1 Introduction

A *general position set* of a graph G is a set of vertices $S \subseteq V(G)$ such that no three vertices from S lie on a common shortest path of G . The order of a largest general position set, shortly called a *gp-set*, is the *general position number* $\text{gp}(G)$ of G (also written gp-number). This concept was recently and independently introduced in [13, 21]. We should mention though that the same concept was studied on hypercubes already in 1995 by Körner [15]. Following [13] and its notation and terminology, the concept received a lot of attention, see the series of papers [7, 10, 11, 12, 14, 17, 20]. In particular, in [17] the general position problem was studied on complementary prisms. In order to characterize an extremal case for the general position number of these graphs, the concept of general 3-position was introduced as an essential ingredient of the characterization. In this paper we extend this idea as follows.

Let $d \in \mathbb{N}$ and let G be a (connected) graph. Then $S \subseteq V(G)$ is a *general d -position set* if the following holds:

$$\{u, v, w\} \in \binom{S}{3}, v \in I_G(u, w) \Rightarrow d_G(u, w) > d, \quad (1)$$

where $d_G(u, w)$ denotes the shortest-path distance in G between u and w , and $I_G(u, w) = \{x \in V(G) : d_G(u, w) = d_G(u, x) + d_G(x, w)\}$ is the *interval* between u and w . In words, S is a general d -position set if no three different vertices from S lie on a common geodesic of length at most d . We will say that vertices u, v, w that fulfill condition (1) lie in *general d -position*. The cardinality of a largest general d -position set in a graph G is the *general d -position number* of G and is denoted by $\text{gp}_d(G)$.

We proceed as follows. In the rest of this section we recall needed definitions and state some basic facts and results on the general d -position number. Then, in Section 2, we demonstrate that in the inequality chain $\text{gp}_{\text{diam}(G)}(G) \leq \text{gp}_{\text{diam}(G)-1}(G) \leq \dots \leq \text{gp}_2(G)$ all kinds of equality and strict inequality cases are possible. Using one of the corresponding constructions we also prove that the problem of determining the gp_d number is NP-complete. In Section 3 we determine the gp_d number of paths and cycles and give a general upper bound on the gp_d number in term of the diameter of a given graph. In the subsequent section we prove a structural characterization of general d -position sets. In Section 5 we report on the connections between general d -position sets and two well-established concepts, the dissociation number and strong resolving graphs. In the concluding section we consider the gp_d number of infinite graphs and pose several open questions.

1.1 Preliminaries

For a positive integer k we will use the notation $[k] = \{1, \dots, k\}$. The *clique number* and the *independence number* of G are denoted by $\omega(G)$ and $\alpha(G)$. If $S \subseteq V(G)$, then the subgraph of G induced by S is denoted by $\langle S \rangle$ and $\binom{S}{k}$ denotes the set of all subsets of S having cardinality k . A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ holds for all $u, v \in V(H)$. If H_1 and H_2 are subgraphs of G , then the distance $d_G(H_1, H_2)$ between H_1 and H_2 is defined as $\min\{d_G(h_1, h_2) : h_1 \in V(H_1), h_2 \in V(H_2)\}$. In particular, if H_1 is the one vertex graph with u being its unique vertex, then we will write $d_G(u, H_2)$ for $d_G(H_1, H_2)$. We say that the subgraphs H_1 and H_2 are *parallel*, denoted by $H_1 \parallel H_2$, if for every pair of vertices $h_1 \in V(H_1)$ and $h_2 \in V(H_2)$ we have $d_G(h_1, h_2) = d_G(H_1, H_2)$. If H_1 and H_2 are not parallel, we will write $H_1 \nparallel H_2$. The *open neighborhood* and the *closed neighborhood* of a vertex v of G will be denoted by $N_G(x)$ and $N_G[x]$, respectively. Vertices x and y of G are *true twins* if $N_G[u] = N_G[v]$. We may omit the subscript G in the above definitions if the graph G is clear from the context.

Clearly, if $d = 1$, then every subset of vertices of G is a general 1-position set, and if $d \geq \text{diam}(G)$, then S is a general d -position set if and only if S is a general position set. Moreover, note that

$$\text{gp}_{\text{diam}(G)}(G) \leq \text{gp}_{\text{diam}(G)-1}(G) \leq \dots \leq \text{gp}_2(G). \quad (2)$$

We conclude the preliminaries with the following useful property.

Proposition 1.1 *Let G be a graph and let $2 \leq d \leq \text{diam}(G) - 1$ be a positive integer. If H_1, \dots, H_r are isometric subgraphs of G such that $d_G(H_i, H_j) \geq d$ for $i \neq j$, then $\text{gp}_d(G) \geq \sum_{i=1}^r \text{gp}_d(H_i)$.*

Proof. For each $i \in [r]$, let S_i be a general d -position set of H_i such that $|S_i| = \text{gp}_d(H_i)$. We claim that $S = \bigcup_{i=1}^r S_i$ is a general d -position set of G . Suppose $\{x, y, z\} \in \binom{S}{3}$ such that $y \in I_G(x, z)$ and $d_G(x, z) \leq d$. That is, there exists a shortest xz -path of length at most d in G that contains y . Since $d_G(u, v) \geq d$ for any two vertices $u \in V(H_i)$ and $v \in V(H_j)$ with $i \neq j$, there exists $k \in [r]$ such that $\{x, y, z\} \subseteq V(H_k)$. Now, since H_k is an isometric subgraph of G , it follows that $d_{H_k}(x, y) = d_G(x, y)$, $d_{H_k}(y, z) = d_G(y, z)$ and $d_{H_k}(x, z) = d_G(x, z)$. This implies that there is a xz -geodesic in H_k that contains y . Hence, $y \in I_{H_k}(x, z)$, and since S_k is a general d -position set of H_k , we infer that $d_G(x, z) = d_{H_k}(x, z) > d$, which is a contradiction. Therefore, S is a general d -position set of G , and it follows that $\text{gp}_d(G) \geq \sum_{i=1}^r \text{gp}_d(H_i)$. \square

2 On the inequality chain (2) and computational complexity

In this section we investigate the inequality chain (2) by constructing different classes of graphs which demonstrate that all kinds of equality and strict inequality cases can happen. We conclude the section by applying one of these constructions to prove that the GENERAL d -POSITION PROBLEM is NP-complete.

Equality in (2) simultaneously.

For $n \geq 2$, let S be a star with center x and leaves $u_1, \dots, u_n, v_1, \dots, v_n$. Construct a graph G_n of order $2n + 3$ by taking the disjoint union of S and an independent set of vertices $\{u, v\}$ together with the set of edges $\{uu_i, vv_i : i \in [n]\}$. The diameter of G_n is 4, and we have $\text{gp}_4(G_n) = \text{gp}_3(G_n) = \text{gp}_2(G_n) = 2n$.

Equality in (2) simultaneously again.

Let T_r , $r \geq 2$, be the tree obtained from P_{r+1} by attaching two leaves to each of its internal vertices. Then we claim that

$$\text{gp}_r(T_r) = \text{gp}_{r-1}(T_r) = \dots = \text{gp}_2(T_r).$$

Indeed, first note that $\text{diam}(T_r) = r$. Since the gp-number of a tree is the number of its leaves (cf. [13, Corollary 3.7]), we have $\text{gp}_r(T_r) = 2r$. Let next S be a general 2-position set. If u is a vertex of T_r adjacent to exactly two leaves, say v and w , then $|S \cap \{u, v, w\}| \leq 2$. Moreover, if u is a vertex of T_r adjacent to exactly three leaves, say v , w , and z , then $|S \cap \{u, v, w, z\}| \leq 3$. It follows that $\text{gp}_2(T_r) \leq 2(r-3) + 2 \cdot 3 = 2r$. In conclusion, $2r = \text{gp}_r(T_r) \leq \text{gp}_{r-1}(T_r) \leq \dots \leq \text{gp}_2(T_r) \leq 2r$, hence equality holds throughout.

Strict inequality in (2) in exactly one case.

Let $k, \ell \geq 4$ and let $G_{k,\ell}$ be a graph defined as follows. Its vertex set is

$$V(G_{k,\ell}) = \bigcup_{j=1}^k \{u_j, w_j, x_{j,1}, \dots, x_{j,\ell}\} \cup \{x_{k+1,1}\}.$$

For $j \in [k]$, each of the vertices u_j and w_j is adjacent to $x_{j,1}, \dots, x_{j,\ell}$ and to $x_{j+1,1}$. There are no other edges in $G_{k,\ell}$. Note that $|V(G_{k,\ell})| = k(\ell + 2) + 1$ and that $\text{diam}(G_{k,\ell}) = 2k$.

It is straightforward to see that the set $X = \bigcup_{j=1}^k \{x_{j,1}, \dots, x_{j,\ell}\} \cup \{x_{k+1,1}\}$ is a largest independent set of $G_{k,\ell}$. Moreover, X is also a largest general 2-position set and a largest general 3-position set. Furthermore, it is not difficult to infer that the

set $X \setminus \{x_{2,1}, \dots, x_{k,1}\}$ is a largest general d -position set for each $d \in \{4, \dots, 2k\}$. In conclusion,

$$\text{gp}_{2k}(G_{k,\ell}) = \text{gp}_{2k-1}(G_{k,\ell}) = \dots = \text{gp}_4(G_{k,\ell}) < \text{gp}_3(G_{k,\ell}) = \text{gp}_2(G_{k,\ell}) = \alpha(G).$$

Strict inequality in (2) in every case.

Given a positive integer t , construct the graph H_t as follows. Begin with a complete graph K_{4t} with vertex set $V(K_{4t}) = A \cup B$ where $|A| = |B| = 2t$. Next, add a path $P_{t-1} = v_1 \dots v_{t-1}$, and join with an edge every vertex of B with the leaf v_1 of P_{t-1} . Then, add a pendant vertex u_i to every vertex $v_i \in \{v_2, \dots, v_{t-1}\}$, and finally, for every $i \in \{2, \dots, t-1\}$, add the edge $u_i v_{i-1}$. As an example, the graph H_8 is represented in Figure 1.

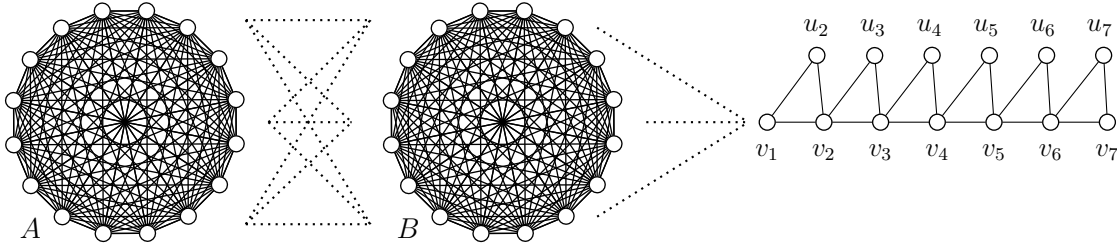


Figure 1: The graph H_8 . Edges joining the sets A and B , as well as joining B with the vertex v_1 are indicated with dotted lines.

Notice that the graph H_t has diameter t . The general d -position number of H_t for all possible d is given in the following result.

Proposition 2.1 *If $2 \leq d \leq t$, then*

$$\text{gp}_d(H_t) = \begin{cases} 4t; & d = t, \\ 4t + 2; & d = t - 1, \\ 5t - d + 1; & \text{otherwise.} \end{cases}$$

Proof. We first note that the set $A \cup B$ is a general position set of H_t , or equivalently a general t -position set. Thus, $\text{gp}(H_t) = \text{gp}_t(H_t) \geq 4t$. Suppose $\text{gp}(H_t) = \text{gp}_t(H_t) > 4t$ and let S be a general t -position set. Hence, there exists at least one vertex not in $A \cup B$ which is in S . Since every shortest path joining a vertex of A with a vertex not in $A \cup B$ passes through a vertex in B , it follows that $S \cap A = \emptyset$ or $S \cap B = \emptyset$. This implies that $|S| \leq 2t + 2t - 3 = 4t - 3$, and this is not possible. Therefore $\text{gp}(H_t) = \text{gp}_t(H_t) = 4t = 5t - d$, when $d = t$.

We next consider the case $d = t - 1$. The set $A \cup B \cup \{v_{t-1}, u_{t-1}\}$ is a general $(t-1)$ -position set of H_t , and so, $\text{gp}_{t-1}(H_t) \geq 4t + 2$. If we suppose that $\text{gp}_{t-1}(H_t) > 4t + 2$, then a similar argument to that above for $d = t$ leads to a contradiction. Therefore, $\text{gp}_{t-1}(H_t) = 4t + 2$.

We finally consider $d = t - k$ with $2 \leq k \leq t - 2$. Notice that the set $A \cup B \cup \{v_{t-1}, u_{t-1}, u_{t-2}, \dots, u_{t-k}\}$ is a general d -position set of H_t of cardinality $4t + k + 1 = 4t + (t - d + 1) = 5t - d + 1$, and so, $\text{gp}_d(H_t) \geq 5t - d + 1$. Again, an argument similar to the two cases above leads to $\text{gp}_d(H_t) = 5t - d + 1$. \square

Proposition 2.1 yields strict inequalities in the chain (2), that is, for any graph H_t with $t \geq 3$, we have

$$\text{gp}_t(H_t) < \text{gp}_{t-1}(H_t) < \dots < \text{gp}_2(H_t). \quad (3)$$

We shall finish this section by considering the computational complexity of the decision problem related to finding the general d -position number of graphs, in which we also show the usefulness of the above graphs H_t .

GENERAL d -POSITION PROBLEM

Input: A graph G , an integer $d \geq 2$, and a positive integer r .
Question: Is $\text{gp}_d(G)$ larger than r ?

We first remark that the GENERAL d -POSITION PROBLEM is known to be NP-complete for every $d \geq \text{diam}(G)$ (see [13]). Hence, we may center our attention on the cases $d \in \{2, \dots, \text{diam}(G) - 1\}$, although our reduction also works for the case $d = \text{diam}(G)$.

Theorem 2.2 *If $d \geq 2$, then the GENERAL d -POSITION PROBLEM is NP-complete.*

Proof. First, we can readily observe that the problem belongs to the class NP, since checking that a given set is indeed a general d -position set can be done in polynomial time. From now on, we make a reduction from the MAXIMUM CLIQUE PROBLEM to the GENERAL d -POSITION PROBLEM.

In order to present the reduction, for a given graph G of order t , we shall construct a graph G' by using the above graphs H_t . We construct G' from the disjoint union of G and H_t , by adding all possible edges between $A \cup B \cup \{v_1\}$ and $V(G)$. It is then easily observed that $\omega(G') = |A| + |B| + \omega(G)$. Moreover, using similar arguments as in the proof of Proposition 2.1, we deduce that $\text{gp}_d(G') = \text{gp}_d(H_t) + \omega(G)$. From this fact, since the value $\text{gp}_d(H_t)$ is known from Proposition 2.1, the reduction is completed, and the theorem is proved. \square

3 Paths and cycles

In this section we determine the general d -position number of paths and cycles. The first result in turn implies a general upper bound on the general d -position number in term of the diameter of a given graph.

Proposition 3.1 *If $n \geq 3$ and $2 \leq d \leq n - 1$, then*

$$\text{gp}_d(P_n) = \begin{cases} 2 \left\lfloor \frac{n}{d+1} \right\rfloor - 1; & n \equiv 1 \pmod{d+1}, \\ 2 \left\lfloor \frac{n}{d+1} \right\rfloor; & \text{otherwise.} \end{cases}$$

Proof. Let $d \in \{2, \dots, n - 1\}$ and $P_n = v_1 v_2 \dots v_n$. If $n \equiv 1 \pmod{d+1}$, then let

$$S = \{v_{(d+1)i+1}, v_{(d+1)i+2} : 0 \leq i \leq \lfloor n/(d+1) \rfloor - 1\} \cup \{v_n\},$$

and if $n \not\equiv 1 \pmod{d+1}$, then let

$$S = \{v_{(d+1)i+1}, v_{(d+1)i+2} : 0 \leq i \leq \lfloor n/(d+1) \rfloor - 1\}.$$

It can be readily seen that S is a general d -position set of P_n , which gives the lower bound

$$\text{gp}_d(P_n) \geq \begin{cases} 2 \left\lfloor \frac{n}{d+1} \right\rfloor - 1; & n \equiv 1 \pmod{d+1}, \\ 2 \left\lfloor \frac{n}{d+1} \right\rfloor; & \text{otherwise.} \end{cases}$$

On the other hand, suppose

$$\text{gp}_d(P_n) > \begin{cases} 2 \left\lfloor \frac{n}{d+1} \right\rfloor - 1; & n \equiv 1 \pmod{d+1}, \\ 2 \left\lfloor \frac{n}{d+1} \right\rfloor; & \text{otherwise,} \end{cases}$$

and let S' be a general d -position set of cardinality $\text{gp}_d(P_n)$. By the pigeonhole principle, we deduce that there exists a subpath in P_n of length d that contains at least three elements of S' , but this is not possible. Therefore, the desired equality follows. \square

Specializing to $n = 14$ in Proposition 3.1, we next show a table with the values of $\text{gp}_d(P_n)$ for every possible value of d . Notice that, equalities and inequalities occur in distinct positions with respect to the chain (2).

The result for paths gives the following general lower bound.

Corollary 3.2 *Let G be a connected graph of diameter d . If $2 \leq k \leq d$, then*

$$\text{gp}_k(G) \geq \begin{cases} 2 \left\lfloor \frac{d+1}{k+1} \right\rfloor - 1; & d \equiv 0 \pmod{k+1}, \\ 2 \left\lfloor \frac{d+1}{k+1} \right\rfloor; & \text{otherwise.} \end{cases}$$

d	2	3	4	5	6	7	8	9	10	11	12	13
$\text{gp}_d(P_{14})$	10	8	6	6	4	4	4	4	4	4	3	2

Table 1: The values of $\text{gp}_d(P_{14})$ for every $2 \leq d \leq 13$.

Proof. Shortest paths are isometric subgraphs; in particular, this holds for diametrical paths. Hence G contains an isometric P_{d+1} , and therefore $\text{gp}_k(G) \geq \text{gp}_k(P_{d+1})$ by Proposition 1.1 with $r = 1$. Applying Proposition 3.1 yields the result. \square

In a similar manner as done for paths, we can compute the general d -position number for cycles. It is easy to show that $\text{gp}_d(C_3) = 3$ for any d , $\text{gp}_1(C_4) = 4$, and $\text{gp}_d(C_4) = 2$ for $d \geq 2$.

Proposition 3.3 *If $n \geq 5$ and $2 \leq d < \lfloor \frac{n}{2} \rfloor$, then*

$$\text{gp}_d(C_n) = \begin{cases} 2 \lfloor \frac{n}{d+1} \rfloor + 1; & n \equiv d \pmod{d+1}, \\ \lfloor 0.2cm \rfloor 2 \lfloor \frac{n}{d+1} \rfloor; & \text{otherwise.} \end{cases}$$

If $d \geq \lfloor \frac{n}{2} \rfloor$, then $\text{gp}_d(C_n) = 3$.

Proof. Let $C_n = v_1 v_2 \dots v_n v_1$. Note that $\text{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$, and the argument naturally splits into two cases.

First assume that $2 \leq d < \lfloor \frac{n}{2} \rfloor$. Let $m = \lfloor \frac{n}{d+1} \rfloor$ and for each $k \in [m]$ we define X_k by $X_k = \{v_i : (k-1)(d+1) + 1 \leq i \leq k(d+1)\}$. Let $X = V(C_n) \setminus \bigcup_{k=1}^m X_k$. Note that $|X| = x$ where $n \equiv x \pmod{d+1}$ and x is the unique integer such that $0 \leq x \leq d$. If $x \neq d$, then let $S = \{v_{(k-1)(d+1)+1}, v_{(k-1)(d+1)+2} : 1 \leq k \leq m\}$. If $x = d$, then let $S = \{v_{(k-1)(d+1)+1}, v_{(k-1)(d+1)+2} : 1 \leq k \leq m\} \cup \{v_{m(d+1)+1}\}$. It is straightforward to check that in both cases S is a general d -position set, which shows that the claimed value is a lower bound for $\text{gp}_d(C_n)$. As in the proof of Proposition 3.1, an application of the pigeonhole principle establishes the upper bound.

Since $\text{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$, to prove the second statement it is sufficient to show that $\text{gp}_d(C_n) = 3$ for $d = \lfloor \frac{n}{2} \rfloor$. For this purpose, let $S = \{v_1, v_3, v_{\lfloor \frac{n}{2} \rfloor + 2}\}$. For $n = 2r$, we see that $S = \{v_1, v_3, v_{r+2}\}$ and $d = r$. On the other hand, for $n = 2r + 1$, we have $S = \{v_1, v_3, v_{r+3}\}$ and $d = r$. In both cases an easy computation shows that none of the three vertices lies on a shortest path in C_n between the other two vertices. Therefore, S is a general d -position set, and it follows that $\text{gp}_d(C_n) \geq 3$. Suppose T is an arbitrary general d -position set of C_n . We may assume without loss of generality that $v_1 \in T$. It follows that $|T \cap \{v_2, \dots, v_{r+1}\}| \leq 1$ and $|T \cap \{v_{r+2}, \dots, v_n\}| \leq 1$, for otherwise T contains three vertices that lie on a path of length at most d . Therefore, $\text{gp}_d(C_n) \leq |T| \leq 3$. \square

4 A characterization of general d -position sets

In [1, Theorem 3.1] a structural characterization of general position sets of a given graph was proved. In this section we give such a characterization for general d -position sets and as a consequence deduce the characterization from [1].

Theorem 4.1 *Let G be a connected graph and let $d \geq 2$ be an integer. Then $S \subseteq V(G)$ is a general d -position set if and only if the following conditions hold:*

- (i) $\langle S \rangle$ is a disjoint union of complete graphs Q_1, \dots, Q_ℓ .
- (ii) If $Q_i \nparallel Q_j$, $i \neq j$, then $d_G(Q_i, Q_j) \geq d$.
- (iii) If $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) = d_G(Q_i, Q_k)$ for $\{i, j, k\} \in \binom{[\ell]}{3}$, then $d_G(Q_i, Q_k) > d$.

Proof. Let S be a general d -position set of G and let H be a connected component of $\langle S \rangle$. If H is not complete, then it contains an induced P_3 . The vertices of this P_3 are on a geodesic of length 2 which is not possible since they belong to S and $d \geq 2$. Hence H must be complete.

Consider next two cliques Q_i and Q_j that are not parallel. Let $d_G(Q_i, Q_j) = p$ and let $u \in Q_i$ and $v \in Q_j$ be vertices with $d_G(u, v) = p$. Since $Q_i \nparallel Q_j$, we may assume without loss of generality that there is a vertex $w \in Q_i$ such that $d_G(w, Q_j) = p + 1$. Then u lies on a w, v -geodesic of length $p + 1$ which implies that $p + 1 \geq d + 1$ and so, $d_G(Q_i, Q_j) = p \geq d$.

Assume next that $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) = d_G(Q_i, Q_k)$ for some $\{i, j, k\} \in \binom{[\ell]}{3}$. If $Q_i \nparallel Q_j$, then by the already proved condition (ii) we immediately get that $d_G(Q_i, Q_j) \geq d$ and thus $d_G(Q_i, Q_k) > d$. The same holds if $Q_j \nparallel Q_k$. Hence assume next that $Q_i \parallel Q_j$ and $Q_j \parallel Q_k$. Let $u \in Q_i$ and $w \in Q_k$ be vertices with $d_G(u, w) = d_G(Q_i, Q_k)$. Since $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) = d_G(Q_i, Q_k)$, $Q_i \parallel Q_j$, and $Q_j \parallel Q_k$, it follows that $d_G(u, w) = d_G(u, v) + d_G(v, w)$ for every vertex v of Q_j . We conclude that $d_G(Q_i, Q_k) > d$.

To prove the converse, assume that conditions (i), (ii), and (iii) are fulfilled for a given set S and let $\{u, v, w\} \in \binom{S}{3}$. We need to show that u, v, w lie in general d -position.

If u, v, w lie in the same connected component of $\langle S \rangle$, then by (i), this component is complete and the assertion is clear. Suppose next that u, v, w lie in the union of cliques Q_i and Q_j . If $Q_i \parallel Q_j$, then u, v, w are clearly in general d -position. And if $Q_i \nparallel Q_j$, then u, v, w lie in general d -position by (ii).

In the last case to consider the three vertices lie in different cliques, say $u \in Q_i$, $v \in Q_j$, and $w \in Q_k$. If the assertion does not hold, then the three vertices lie on

a common geodesic and we may assume without loss of generality that $d_G(u, w) = d_G(u, v) + d_G(v, w)$. If $Q_i \nparallel Q_j$, then by (ii), we get $d_G(Q_i, Q_j) \geq d$ and hence $d_G(u, w) = d_G(u, v) + d_G(v, w) \geq d_G(Q_i, Q_j) + d_G(Q_j, Q_k) \geq d + 1 > d$. Analogously, if $Q_j \nparallel Q_k$, we also get $d_G(u, w) > d$. Suppose then that $Q_i \parallel Q_j$ and $Q_j \parallel Q_k$. If also $Q_i \parallel Q_k$, then $d_G(u, w) = d_G(u, v) + d_G(v, w)$ implies that $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) = d_G(Q_i, Q_k)$ and so $d_G(Q_i, Q_k) > d$ by (iii). Again using the fact that $Q_i \parallel Q_k$, it follows that $d_G(u, w) > d$. We are left with the case that $Q_i \parallel Q_j$, $Q_j \parallel Q_k$, and $Q_i \nparallel Q_k$. If $d_G(u, w) = d_G(Q_i, Q_k)$, then by (iii), we get that $d_G(u, w) > d$. Otherwise we may assume without loss of generality that there exists a vertex $u' \in Q_i$, $u' \neq u$, such that $d_G(Q_i, Q_k) = d_G(u', Q_k) < d_G(u, w)$. Since $Q_i \nparallel Q_k$, (ii) implies that $d_G(u', Q_k) \geq d$. But then $d_G(u, w) > d_G(Q_i, Q_k) \geq d$. \square

Corollary 4.2 [1, Theorem 3.1] *Let G be a connected graph. Then $S \subseteq V(G)$ is a general position set if and only if the following conditions hold:*

- (i) $\langle S \rangle$ is a disjoint union of complete graphs Q_1, \dots, Q_ℓ .
- (ii) $Q_i \parallel Q_j$ for every $i \neq j$.
- (iii) $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) \neq d_G(Q_i, Q_k)$ for every $\{i, j, k\} \in \binom{[\ell]}{3}$.

Proof. Set $d = \text{diam}(G)$, so that general d -position sets are precisely general position sets. Condition (ii) of Theorem 4.1 implies that in cliques Q_i and Q_j , which are not parallel, we can find a pair of vertices at distance larger than $\text{diam}(G)$. Since this is not possible, every two cliques must be parallel. Similarly, if the assumption of condition (iii) would be fulfilled for some cliques Q_i , Q_j , and Q_k , then we would again have vertices at distance larger than $\text{diam}(G)$. Therefore, $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) \neq d_G(Q_i, Q_k)$ must hold for every $\{i, j, k\} \in \binom{[\ell]}{3}$. \square

5 Connections with other topics

In this section we connect general d -position sets with the dissociation number and with strong resolving graphs.

Strong resolving graphs

A vertex u of a connected graph G is *maximally distant* from a vertex v if every $w \in N(u)$ satisfies $d_G(v, w) \leq d_G(u, v)$. If u is maximally distant from v , and v is maximally distant from u , then u and v are *mutually maximally distant* (MMD for

short). Given an integer $d \geq 2$, the *strong d -resolving graph* G_{SR}^d of G has vertex set $V(G)$, and two vertices u, v are adjacent in G_{SR}^d if either u, v are MMD in G , or $d_G(u, v) \geq d$. The terminology used in this construction comes from the notion of the strong resolving graph introduced in [19] as a tool to study the strong metric dimension of graphs. See also [16].

The following observation will be useful in the proof of Theorem 5.2.

Observation 5.1 *If G is connected and a vertex u of G is maximally distant from a vertex v of G , then $u \notin I(v, w)$ for every $w \in V(G) \setminus \{u\}$.*

Proof. For the sake of contradiction suppose there exists such a vertex $w \in V(G) \setminus \{u\}$ such that $u \in I(v, w)$. Suppose that $v = v_0 \dots v_{i-1}u = v_i v_{i+1} \dots v_k = w$ is a v, w -geodesic. Since this is a geodesic, it follows that $d(v, u) = i$. But u is maximally distant from v , and thus $d(v, v_{i+1}) \leq d(v, u) = i$. Now, by following a shortest v, v_{i+1} -path with the path $v_{i+2} \dots v_k = w$ we arrive at a v, w -path of length less than k , which is a contradiction. \square

From Observation 5.1 it follows immediately that if three vertices x, y, z are pairwise MMD, then $x \notin I(y, z)$, $y \notin I(x, z)$, and $z \notin I(x, y)$. From this we infer that x, y, z lie in general d -position.

Theorem 5.2 *If G is a connected graph and $d \geq 2$ is an integer, then $\text{gp}_d(G) \geq \omega(G_{\text{SR}}^d)$.*

Proof. We consider a set $S \subseteq V(G_{\text{SR}}^d)$ that induces a (largest) complete subgraph of G_{SR}^d . Then every two vertices $x, y \in S$ are MMD in G , or $d_G(x, y) \geq d$. We now consider three vertices x, y, z of S in the graph G . If they are pairwise MMD in G , then as above, x, y, z lie in general d -position. Suppose then that two of them, say x and y , are not MMD in G . Since x, y are adjacent in G_{SR}^d , it follows that $d_G(x, y) \geq d$. Suppose for instance that x, z are MMD in G . By Observation 5.1, it follows that $x \notin I(z, y)$ and $z \notin I(x, y)$. If $y \in I(x, z)$, then $d_G(x, z) = d_G(x, y) + d_G(y, z) \geq d + 1$, and hence x, y, z lie in general d -position. On the other hand, if $y \notin I(x, z)$, then by definition, x, y, z lie in general d -position.

It remains only to consider the case in which no pair of x, y, z is MMD in G . This means that the distance between any two of them is at least d , and this clearly means that x, y, z are in general d -position. \square

Note that if $d = \text{diam}(G)$, then G_{SR}^d is the standard strong resolving graph G_{SR} as defined in [19]. In this case Theorem 5.2 reduces to $\text{gp}(G) \geq \omega(G_{\text{SR}})$, a result earlier obtained in [12, Theorem 3.1].

Dissociation number and independence number

If G is a graph and $S \subseteq V(G)$, then S is a *dissociation set* if $\langle S \rangle$ has maximum degree at most 1. The *dissociation number* $\text{diss}(G)$ of G is the cardinality of a largest dissociation set in G . This concept was introduced by Yanakkakis [22]; see also [3, 4, 9]. Further, a *k-path vertex cover* of G is a subset S of vertices of G such that every path of order k in G contains at least one vertex from S . The minimum cardinality of a *k-path vertex cover* in G is denoted by $\psi_k(G)$. The minimum 3-path vertex cover is a dual problem to the dissociation number because $\text{diss}(G) = |V(G)| - \psi_3(G)$; see [9]. For the algorithmic state of the art on the 3-path vertex cover problem see [2].

Proposition 5.3 *If G is a triangle-free graph, $d \geq 2$, and $S \subseteq V(G)$ is a general d -position set, then S is a dissociation set. Moreover, if $d = 2$, then S is a general 2-position set if and only if S is a dissociation set.*

Proof. Let d be a positive integer such that $d \geq 2$. Suppose that S is a general d -position set in a triangle-free graph G . By Theorem 4.1 every component of the subgraph $\langle S \rangle$ of G induced by S is a complete graph. Since G is triangle-free, we conclude that each of these components has order 1 or 2. Therefore, S is a dissociation set. Now assume that $d = 2$ and S is a dissociation set in G . The components C_1, \dots, C_k of $\langle S \rangle$ each have order 1 or 2 and are thus complete graphs. For every pair of distinct indices i, j in $[k]$, the fact that C_i and C_j are distinct components of the induced subgraph $\langle S \rangle$ implies that $d_G(C_i, C_j) \geq 2$. Therefore, conditions (ii) and (iii) of Theorem 4.1 follow immediately, and hence S is a general 2-position set. \square

Proposition 5.3 immediately gives the following result for triangle-free graphs.

Corollary 5.4 *If G is a triangle-free graph and $d \geq 2$, then $\text{gp}_d(G) \leq \text{diss}(G)$. Moreover, $\text{gp}_2(G) = \text{diss}(G)$.*

We next relate the particular case of general 2-position number with the independence number of graphs.

Proposition 5.5 *If G is a connected graph without true twins, then $\text{gp}_2(G) \geq \alpha(G)$.*

Proof. Let $x, y \in V(G)$. Suppose first that $xy \in E(G)$. Since x and y are not true twins, it follows that x and y are not MMD. By definition, we infer that $xy \notin E(G_{\text{SR}}^2)$. On the other hand, if $xy \notin E(G)$, then $d_G(x, y) \geq 2$ and by definition $xy \in E(G_{\text{SR}}^2)$. Consequently, G_{SR}^2 is the complement \overline{G} of G . Then by using Theorem 5.2, we have $\text{gp}_2(G) \geq \omega(\overline{G}) = \alpha(G)$. \square

It is straightforward to see that if $2 \leq m \leq n$, then $\text{gp}_2(K_{m,n}) = n = \alpha(K_{m,n})$. Hence the bound of Proposition 5.5 is sharp. For another such family consider the grid graphs $P_{2r} \square P_{2s}$. (For the definition of the Cartesian product operation \square see, for instance, [8].) As already mentioned, $\psi_3(G) = n - \text{diss}(G)$ holds for any graph G of order n . Also, from [5] it is known that $\psi_3(P_{2r} \square P_{2s}) = 2rs$. Moreover, from Corollary 5.4, we have that $\text{gp}_2(P_{2r} \square P_{2s}) = \text{diss}(P_{2r} \square P_{2s})$. Thus,

$$\text{gp}_2(P_{2r} \square P_{2s}) = \text{diss}(P_{2r} \square P_{2s}) = 4rs - \psi_3(P_{2r} \square P_{2s}) = 2rs = \alpha(P_{2r} \square P_{2s}).$$

6 Infinite graphs and some open problems

The general position problem has been partially studied also on infinite graphs. In [14] it was proved that $\text{gp}(P_\infty^2) = 4$, where P_∞^2 is the 2-dimensional grid graph (alias the Cartesian product of two copies of the two way infinite path). The general position number of the 2-dimensional strong grid graph was also determined, and it was shown that $10 \leq \text{gp}(P_\infty^3) \leq 16$. In [10] the latter lower bound was improved to 14. All these efforts were recently rounded off in [11] where it is proved that if $n \in \mathbb{N}$, then $\text{gp}(P_\infty^n) = 2^{2^{n-1}}$. On the other hand, the following result reduces the study of the general d -position number of infinite graphs to the case $d = \infty$.

Proposition 6.1 *If G is an infinite graph and $d < \infty$, then $\text{gp}_d(G) = \infty$.*

Proof. Let $d < \infty$ be a fixed positive integer. There is nothing to be proved if $d = 1$, hence assume that $d \geq 2$.

Suppose first that $\text{diam}(G) = \infty$. In this case G contains an infinite isometric path $P = v_1 v_2 \dots$. It is clear that $\{v_{di} : i \in \mathbb{N}\}$ is a general d -position set, and hence $\text{gp}_d(G) = \infty$.

Suppose second that $\text{diam}(G) < \infty$. Considering an arbitrary vertex of G and its distance levels we infer that G contains a vertex x with $\deg(x) = \infty$. Let $H = \langle N[x] \rangle$. Since H is an infinite graph, Erdős-Dushnik-Miller theorem [6] implies that H contains a (countably) infinite independent set I or an infinite clique Q (of the same cardinality as H). If H contains Q , then Q is also a clique of G , and hence G contains an infinite general d -position set. On the other hand, if H contains I , then I is also an independent set of G . Moreover, having in mind that $H = \langle N[x] \rangle$, we infer that each pair of vertices of I is at distance 2 in G . This fact in turn implies that I is an infinite general d -position set of G . We conclude that $\text{gp}_d(G) = \infty$. \square

6.1 Open questions

In this section we point out several questions that, in our opinion, are worthy of consideration.

- In [18, Lemma 5.1] there is a polynomial algorithm for the dissociation number of trees T and hence for $\text{gp}_2(T)$. On the other hand, $\text{gp}_{\text{diam}(T)}(T)$ can also be efficiently computed. Hence, is it possible to compute in polynomial time $\text{gp}_d(T)$ for any $2 < d < \text{diam}(T)$? More generally, what can be done for the case of block graphs? We know that the simplicial vertices of a block graph form a gp -set. Can the algorithm of Papadimitriou and Yannakakis be modified for block graphs?
- Compare $\text{diss}(G)$ with $\text{gp}_2(G)$ for graphs G with $\omega(G) \geq 3$. Our guess is that these invariants are incomparable in such graphs. Is there some relationship when G is a block graph?
- What is $\text{gp}_d(G)$ whenever G is a grid-like graph? Note that by applying Corollary 5.4 together with Theorem 4.1 in [5], one can find the value of $\text{gp}_2(P_n \square P_m)$ for any n and m . Find $\text{gp}_d(P_n \square P_m)$ for $d \geq 3$. Find the general d -position number of a partial grid graph for $d \geq 2$.

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